ANALYTICAL SOLUTIONS

Method Indicator		
Bottom-Up	Hybrid	Top-Down
		YES

Summary of key issues

Issue	Description
Description	Characterisation of the estuary system or estuary processes into manageable stand alone mathematical equations.
Temporal Applicability	Short term (several tides) to long term (100 years)
Spatial Applicability	Whole estuary or large areas of the estuary
Links with Other Tools	• Asymmetry relationships form a subset of analytical methods; • Always useful as an aid to conceptual understanding alongside any tool.
Data Sources	Sources vary enormously depending on the type of analytical equation
Necessary Software Tools / Skills	• An understanding of physical process; • (Preferable) A familiarity with mathematical argument; • Geomorphological interpretation of output.
Typical Analyses	• As a conceptual aid to understanding how the system will evolve following some change; • As a back-of-the envelope quantitative assessment.
Limitations	It can be necessary to consider the mathematics of the method in detail to understand whether a particular analytical method can reliably be applied to a given estuary system.

Analytical solutions are a group of mathematical expressions, often derived from basic physical principles, but usually resulting from a simplification of estuary systems that can be utilised to gain insight into the functioning and potential changes within an estuary system.

Overview of technique

Analytical solutions exist for a diverse range of physical processes and mechanisms encountered within estuarine environments. These include tidal propagation and current flow, residual circulation, saline intrusion, wind-wave generation, wave propagation and evolution, sediment transport, flocculation and contaminant mixing.

The role of analytical solutions is to simplify the estuary system or the physical process in question into something more tangible and useable. As a result, the complex and often random nature of natural systems, which obscures the underlying process, can be removed to reveal the first order relationships associated with the system or physical process. These relationships then allow a straightforward evaluation of the impact of changes to the underlying drivers in the system or physical process to be evaluated. Usually this simplification enables changes to estuary systems to be evaluated in a qualitative manner, but in appropriate circumstances analytical solutions can provide quantitative estimates. In such circumstances the simplified nature of analytical solutions typically results in tools which are quick to use and require only a calculator or spreadsheet to implement.

In order to derive the analytical solutions, a number of simplifying assumptions normally have to be made regarding the estuary system or physical process. The determination of these assumptions requires judgement and experience to assess whether the analytical solution can be applied because the assumptions must be appropriate for a given situation.

The scope of analytical methods that could be considered is wide and therefore it is has been necessary to limit those considered here to those that featured in the Estuaries Research Programme Phase 1 study (EMPHASYS, 2000) and the Phase 2 EstProc study (EstProc Consortium, 2006), which include:

- Tidal flow;
- Residual current profiles and saline intrusion;
- Residual sediment transport; and
- Suspended sediment concentration profiles.

For each of these an introduction to the theory is presented and any applicable approaches, followed by conclusions. This is in turn followed by best practice advice for each.

Application to tidal flow - theory

Introduction

The classical representation of estuaries is that of a one-dimensional linearised governing equation for water level variation, η, in a prismatic frictionless channel, which reduces to the familiar second-order wave equation:

$$\frac{\partial^2 \eta}{\partial t^2} = c_0^2 \frac{\partial^2 \eta}{\partial x^2} \tag{1}$$

where: $c_0 = (gH)^{1/2}$ and H is the (mean) water depth.

For a sinusoidally forced channel closed at one end Equation 1 produces a standing wave solution, characterised by incident and reflected waves causing tidal amplitude to vary through nodes and anti-nodes and producing a relative phase of 90º between velocity and water level (i.e. LW slack occurs roughly at LW, HW slack occurs roughly at HW and peak tides occur roughly at mean water level). If the channel has a length of exactly one quarter the (tidal) wavelength then the incident and reflected waves cancel entirely at the mouth and resonance occurs within the channel. In a channel of infinite length Equation 1 produces a progressive wave with a relative phase of 0º (since there is no reflected wave) and the phase speed becomes equal to c_0.

In most real UK estuaries the tidal phase speed has been observed to be close to c_0 and the relative phase is observed to be close to 90º. Hunt (1964) showed that this behaviour (based on the Thames Estuary) was inconsistent with Equation 1 and more consistent with the effect of friction in “strongly convergent” channels (i.e. channels where the proportional change in cross-section area with distance along the estuary is much larger than the proportional change in peak current speed). He de-emphasized the importance of incident and reflected waves and instead expressed solutions as propagating wave forms modified by geometry and friction.

Since this ground breaking work of Hunt (1964) various approaches have been tried which have included different mathematical terms, and to different order, in the form of the momentum and continuity equations. Friedrichs and Aubrey (1994) tried to bring some rigour to this problem by formally evaluating which terms were important and at which orders of accuracy. It should be noted however, that the form of the solution given by Equation 1 (i.e. frictionless channel with constant width and depth) is still often used but in most cases (at least in the UK) will be a poor representation of estuary tidal propagation.

Pillsbury's approach

The focus will now be on the form of the 1D hydrodynamic equations in Equations 2 and 3:

$$B\frac{\partial \eta}{\partial t}+\frac{\partial\{Au\}}{\partial x}=0 \quad \text{(continuity)} \qquad (2)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=-g\frac{\partial \eta}{\partial x}-\frac{fu|u|}{H}=0 \quad \text{(momentum)} \qquad (3)$$

where: *u* is the cross-section averaged current speed;
B is the channel width;
A is the cross-section area;
η is the water level above mean sea level;
H is the water depth;
f is the friction coefficient.

Possibly the most simple form of this is the "ideal" estuary solution developed by Pillsbury (1956), who assumed the following:

- Rectangular cross-section;
- Width reduces exponentially with distance upstream,;
- Depth is constant;
- Tidal range is constant,
- The $u\frac{\partial u}{\partial x}$ term (the 2nd term on the LHS of Equation 3) is ignored.

$$\eta = a.\sin(\sigma t - kx) \qquad (4)$$

$$u=\frac{ag}{c}\sin\phi\sin(\sigma t-kx-\phi) \qquad (5)$$

where: the exponential variation in B is given by:

$$B = B_0\, exp\{-kx\cot\phi\} \qquad (6)$$

and $c=\frac{\sigma}{k}=\sqrt{gH}$ (7)

where: η is the water level above mean sea level;
a is the amplitude of the water level variation;
σ is the tidal frequency;
u is the cross-section averaged current speed;
ϕ is the phase lag of u relative to η;
x is the distance upstream from the estuary mouth;
B is the width of the cross-section;
B_0 *is* the width of the estuary mouth;
k is the wave number;
H is the mean water depth;
g is the acceleration due to gravity.

Friedrichs and Aubrey's approach

Friedrichs and Aubrey (1994) give the 1D estuary solution in a more generalised form for the slightly different configuration of an estuary cross-section. They assumed that an estuary cross-section is made up of storage and channel components (Figure 1) and computed the 1st and 2nd order solutions for an estuary with exponentially decaying storage width and channel cross-section area.

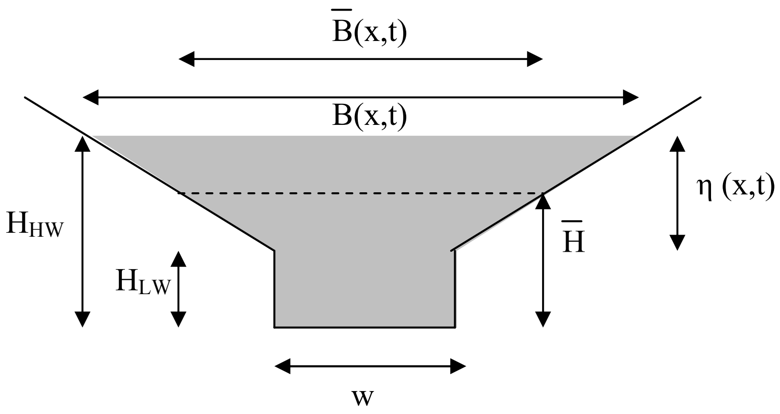

Figure 1. Friedrichs and Aubreys schematisation

Friedrichs and Aubrey assumed the following:

- Width and depth reduce exponentially with distance upstream;
- Tidal range is constant (for 1st order solution), i.e. A.dA/dx >> η.dη/dx;
- The $u\frac{\partial u}{\partial x}$ (the 2nd term on the LHS of Equation 3) can be ignored.

This was shown explicitly to be true where (a) the variation in cross-section area along the estuary is much larger than that of tidal amplitude and (b) the phase speed is of the same order as the frictionless wave speed. The first order continuity and momentum equations were presented as follows:

$$\bar{B}\frac{\partial \eta}{\partial t} = \frac{\bar{A}u}{L_A} \text{ (continuity)} \qquad (8)$$

$$0 = -g\frac{\partial \eta}{\partial x} - Fu \text{ (momentum)} \qquad (9)$$

where: F is given by $F = \frac{8}{3\pi}\frac{c_d\hat{u}}{\bar{H}}$;

L_A is given by $A = A_0 e^{-\frac{x}{L_A}}$;

C_d is the friction coefficient;

$\hat{u}$ is the peak current speed and overbars indicate time-averaged quantities.

The first order solution is given by:

$$\eta = a\cos(\sigma t - kx) \quad \text{where } c = \frac{\sigma}{k} = \frac{c_0^2}{FL_A} \tag{10}$$

$$u = \hat{u}\sin(\sigma t - kx) \quad \text{where } \hat{u} = \frac{aL_A\sigma}{\left(\bar{A}/\bar{B}\right)} \tag{11}$$

Equations 4 and 5, or 10 and 11, can be used to give an indication of the manner in which the hydrodynamics in an estuary will initially change following a change in bathymetry. For example, an increase in depth will give rise to a reduction in current speed and a change in the growth or reduction of the tidal range and current speed. Over the longer term there will be a morphological response to the change in hydrodynamics. In particular sediment will tend to diffuse away from current maxima to areas where currents are smaller (Friedrichs and Aubrey, 1994).

Prandle's approach

Prandle composed a solution to the 1D estuary problem (Prandle and Lane, 2003; Prandle, 2003, 2004b), in which he assumed a constant estuary depth to width ratio and a triangular cross-section shape and (as for the methods of Pillsbury and of Friedrichs and Aubrey) that the acceleration terms can be ignored. In this case Prandle derived the following form of Equations 2 and 3;

$$\frac{\partial\eta}{\partial t} + u\frac{\partial D}{\partial x} + \frac{D}{2}\frac{\partial u}{\partial x} = 0 \tag{12}$$

$$\frac{\partial u}{\partial t} + g\frac{\partial\eta}{\partial x} + Fu = 0 \quad \text{where } F = \frac{8}{3\pi}(e-1)\frac{f\hat{u}}{D} = 1.46\frac{f\hat{u}}{D} \tag{13}$$

and f is a constant of order O(2.5x10^{-3}).

This gave rise to the solutions:

$$\eta = a.\cos(\sigma t - kx) \tag{14}$$

$$u = \hat{u}\cos(\sigma t - kx + \theta) \tag{15}$$

where:

$$\hat{u} = \frac{agk}{\left(\sigma^2 + F^2\right)^{1/2}} \tag{16}$$

$$k = \frac{\sigma}{\left(\frac{Dg}{2}\right)^{1/2}} \tag{17}$$

$$\tan\theta = -\frac{F}{\sigma} = \frac{2S}{Dk} \tag{18}$$

and where: $\hat{u}$ is the current amplitude, and $S = \frac{\partial D}{\partial x}$.

Using Equations 16 to 18 Prandle solved for S (∂D/∂x) and, by integrating S, deduced the values of depth (and therefore width) along the estuary.

For the case where F>>σ the solution for depth D (and therefore width) can be solved analytically to give:

$$D = \left[\frac{5}{4} \frac{\left(a_L 1.46 f\sigma\right)^{1/2}}{\left(2g\right)^{1/4}} \right]^{4/5} x'^{4/5} \qquad (19)$$

where: x' = L –x ; and
L is the length of the estuary (defined as the point where depth becomes zero).

On the basis of Equation 19 the width and depth in an estuary vary as (or close to) $D=D_0[^{(L-x)}/_L]^{4/5}$, $W=W_0[^{(L-x)}/_L]^{4/5}$. The definition of F (Equation 13) and Equations 16 and 17 then infer that $\hat{u}$ is proportional to $[^{(L-x)}/_L]^{1/5}$.

Discussion

The solutions of Pillsbury relate to constant tidal amplitude and depth and exponentially varying width, which results in a constant peak velocity with distance landward; those of Friedrichs and Aubrey relate to constant tidal amplitude and exponentially varying width and depth which result in an exponentially varying velocity with distance landward; while Prandle composes a solution with a constant width/depth ratio and a constant tidal amplitude and a velocity, depth and (since width varies linearly with depth) width, that are interrelated via Equations 16 to 18.

When computing the hydrodynamics it is usual to assume some simple characterisation of the bed (such as depth is constant or varying exponentially etc.) from which the landward variation in velocity is then computed as a result. However, Prandle chooses to calculate the variation in depth (and therefore width) directly from Equations 16 to 18. As a result Prandle is calculating a property, D, that is normally considered as a boundary condition of the problem. If the variation in cross-section area along the estuary is correct then the prediction of along estuary (cross-section-averaged) current velocity will also be approximately correct, i.e. correct to first order (Friedrichs and Aubrey, 1994).

Equation 19 suggests that all estuaries, whose longitudinal variation in currents and tidal amplitude can be said to be roughly constant, will take a specific bathymetric/hydrodynamic form where depth and width are inversely proportional to the four-fifths power of distance from the mouth and peak current speed is inversely proportional to the fifth power of distance from the mouth. Many estuaries vary in the rate of convergence of the bathymetry towards the head and also in their longitudinal changes in current velocity, and hence the analytical methods can only be considered approximate in these situations.

The analytical solutions of Pillsbury and Friedrichs/Aubrey can be used to deduce the effects of changes in estuary bathymetry on tidal flows. In basic terms, assuming tidal amplitude remains unaffected an increase in the rate of decrease of width with distance along the estuary will result in a reduction in current speed while an increase in the rate of decrease of depth with distance along the estuary will result in an increase in current speed.

Conclusions

Based on certain assumptions which have been stated above:

- Analytical formulations such as those of Friedrichs and Aubrey can serve to illustrate the basic functioning of a estuary;
- The analytical solution of Prandle generates information on the idealised form of the estuary.

Residual current profiles and saline intrusion – theory

The non-time-varying version of the 1D momentum equation can be expressed as:

$$g\frac{\partial \eta}{\partial x} + g(\eta - Z)S_x = E_z \frac{\partial^2 u}{\partial Z^2} + E_y \frac{\partial^2 u}{\partial Y^2} \qquad (20)$$

where: g is the acceleration due to gravity;
η is the free surface elevation above mean sea level;
Z is the vertical position;
Y is the horizontal distance from the estuary axis lateral to the direction of flow;
S_x is the density gradient ($\rho^{-1}\partial\rho/\partial x$);
u is the residual current speed at height z and lateral distance y;
E_z & E_y are the vertical and transverse eddy viscosities.

The terms on the right hand side of Equation 20, $E_z \frac{\partial^2 u}{\partial Z^2}$ and $E_y \frac{\partial^2 u}{\partial Y^2}$ represent the effects of vertical and lateral mixing, respectively. Hansen and Rattray (1965, 1966) and Ippen and Harleman (1961) devised systems of estuary classification on the basis that vertical mixing predominates in estuary systems. Others such as Officer (1976) and Prandle (1985) have used the same assumption to derive solutions for the saline-induced residual circulation and intrusion (see below). In the 1970s however Fischer (e.g. Fischer *et al*., 1979), showed that, as in riverine systems, lateral or transverse mixing dominates in well-mixed estuaries with variable depth across their cross-section. Since the work of Fischer, increased attention has been focused upon the dependence of the transverse mixing structure on bathymetry (Friedrichs, 2004), e.g. Nunes and Simpson (1986), Li and Valle-Levinson (1999), Friedrichs and de Velasco (1998), Geyer *et al*. (2000), Mied *et al*. (2002) and Winant and de Velasco (2002). The work of all these authors appears to suggest that stratified or narrow partially mixed estuaries vertical mixing predominates, while in well-mixed estuaries or partially mixed estuaries with variable depth across the cross-section transverse mixing will predominate. This conclusion is supported by measurements from real estuaries (e.g. Murray *et al*., 1975; Dyer, 1974).

For a well-mixed estuary with a longitudinal density gradient and a depth that was homogeneous over the width of the channel Prandle (1985; 2004b) expressed the non-time varying residual current in terms of Equation 20 but assumed the effect of lateral mixing, $E_y \frac{\partial^2 u}{\partial Y^2}$, was negligible. By assuming a density gradient, S_x, constant in time and space and a constant eddy viscosity equal to $E_z = k\hat{u}H$ (with k=0.0025, $\hat{u}$ equal to the tidal amplitude and H, depth) Prandle derived the results:

$$u_s = gS_x \frac{H^3}{E_Z}\left\{-\frac{z^3}{6} + 0.2687z^2 - 0.0373z - 0.0293\right\}$$
$$\approx gS_x \frac{H^2}{k\hat{u}}\left\{-\frac{z^3}{6} + 0.2687z^2 - 0.0373z - 0.0293\right\} \quad (21)$$

where: u_s is the "average" residual near bed current speed along the length of the saline intrusion due to gravitational circulation;
H is the "average" water depth along the length of the saline intrusion;
z s the position above the bed; and
S_x is the longitudinal saline gradient.

This leads to a residual current at the bed given by:

$$u_s = 0.029 gS_x \frac{H^2}{k\hat{u}} \quad (22)$$

Equations 21 and 22 are valid for well-mixed estuaries with homogenous depth over the cross-section. For partially-mixed or stratified estuaries the freshwater input is confined to a surface layer and there exists a saline layer near the bed. Prandle (1985) developed the corresponding solution for residual current at the bed for partially-mixed or stratified estuaries as follows:

$$u_s = \frac{0.018\bar{u}}{1-d} \quad (23)$$

where: ū (= Q_f/A) is the river discharge divided by the cross-section area (the depth-averaged residual velocity); and
d (=D/H) is the thickness of the saline bottom layer divided by the water depth.

The validation of Equations 21 to 23 was undertaken in flume experiments and using data from the Rotterdam Waterway (Prandle, 1985), i.e. in an estuary situation with no lateral bathymetric variation.

Prandle then developed a simple parameterisation for the length of saline intrusion. He proposed a conceptual two layer model with a residual discharge equal to the freshwater flow input flowing in the top layer and zero net residual discharge in the lower layer. Based on an assumption of a linearly varying saline wedge profile he developed the equation:

$$L_i = \frac{0.005H^2}{f\hat{u}u_0}\frac{\Delta\rho}{\rho} \quad (24)$$

where: L_i is the intrusion length,
f is the friction coefficient in Equation 3;
$\hat{u}$ is the "average" peak current speed along the length of the saline intrusion;
u_0 is the velocity resulting from the river discharge divided by H;
ρ & Δρ are the density of sea water and the difference in density between the freshwater river flow and the seawater of the marine boundary.

In a channel of varying cross-section, depth the mixing is dominated by transverse mixing. Fischer *et al*. (1979) give a rough estimate for the ratio, R, of the mixing caused by the transverse gradient to that of the vertical gradient:

$$R \approx \frac{W^2/\varepsilon_t}{H^2/\varepsilon_v} \tag{25}$$

where: W is the channel width and ε_t and ε_v are the transverse and vertical mixing coefficients. Fischer *et al.* (1979) asserted that in all but the most strongly stratified estuaries this ratio can be large. For an estuary of triangular cross-section (of width W and depth H), it can be shown that the value of u_s in Equation 22 becomes,

$$u_s\big|_{bed} = 0.0293 g S_x \frac{H^3}{E_Z}\left(\frac{H^2/E_z}{W^2/E_v}\right) \tag{26}$$

The corollary of this result is that, where such lateral variation in the cross-section exists, the residual longitudinal current will be significantly lower than the result for a narrow partially mixed estuary.

Conclusions

From this assessment of analytical approaches to residual current profiles and saline mixing the following conclusions have been made:

- The longitudinal dispersion of salinity in stratified or narrow partially mixed estuaries is principally a function of vertical mixing, while in well-mixed estuaries or partially mixed estuaries with variable depth across the cross-section transverse mixing will predominate.
- Analytical solutions such as those presented above help to evaluate the residual current velocity caused by a longitudinal salinity gradient, but care must be taken to choose an appropriate solution for specific estuarine condition.

Residual sediment transport - theory

Abbott (1960a) considered a 2DV schematisation of an estuary that is homogenous through the channel width and that varies only gradually in width with distance along the estuary (i.e. not strongly convergent). The tidal variation of the estuary was assumed to be sinusoidal and the eddy viscosity was assumed to be constant through depth. The constituent equations are solved by a method of successive approximation. The second approximation to the longitudinal velocity component takes account of the non-linear convective acceleration terms and contributes a non-periodic term which is in general non-zero. Thus, superimposed on the periodic motion of the water, there is a residual net motion of water near the bed which is indicative of the residual direction of motion. Abbott found that, if the surface velocity in a well-mixed estuary can be described by:

$$U(x,t) = U_0(x)\{\cos wt - \psi(x)\} \tag{27}$$

then, a criterion for landward or seaward transport in the estuary is that:

$$\frac{d}{dx}\left(U_0 e^{\psi}\right) > 0 \Rightarrow \text{landward transport}$$

$$\frac{d}{dx}\left(U_0 e^{\psi}\right) < 0 \Rightarrow \text{seaward transport}$$

Abbott tried this conclusion on the example of the Thames and found that the criterion above could identify the rough location of the turbidity maximum located in the area of the estuary known as the Mud Reaches. However, to some extent this may be a fortunate result since the Thames Estuary is "strongly convergent" and Abbott's analysis is based on estuaries which are constant (or "prismatic") in width and depth (Abbott, 1960a).

It is concluded here that for strongly convergent estuaries like the Thames that Abbott's analysis may still have some use but cannot be relied upon with confidence. For estuary channels which vary more gradually the analysis will be more reliable but it is then also likely that other mechanisms such as salinity-induced density gradients will be important.

In Abbott's second paper (1960b) he considered the magnitude of the various terms in the 2DV longitudinal momentum equation and ignoring these, including the convective term, Abbott derived the criterion,

$$\frac{1}{2}h\left(-\frac{\partial \rho}{\partial x}\right) > \rho\frac{dS}{dx} \Rightarrow \text{landward transport}$$

$$\frac{1}{2}h\left(-\frac{\partial \rho}{\partial x}\right) > \rho\frac{dS}{dx} \Leftarrow \text{seaward transport}$$

This conclusion was tested on the Thames, but the criterion above did not successively identify the turbidity maximum since the density gradient in the Thames (based on measurements from Inglis and Allen, 1957) is (apparently) not able to sustain a landward gradient near the bed. A further test using the example of the Mersey Estuary, which has a stronger longitudinal density gradient, was able to locate the node of zero residual current identified through physical modelling.

Abbott noted that in estuaries such as the Thames where the balance of forces is not simply between the pressure and density gradients, that the convective term becomes important and the criterion above becomes:

$$\frac{1}{2}h\left(-\frac{\partial \rho}{\partial x}\right) > \left(\rho\frac{dS}{dx} + \frac{U_0}{g}\frac{\partial U_0}{\partial x}\right) \Rightarrow \text{landward transport}$$

$$\frac{1}{2}h\left(-\frac{\partial \rho}{\partial x}\right) < \left(\rho\frac{dS}{dx} + \frac{U_0}{g}\frac{\partial U_0}{\partial x}\right) \Rightarrow \text{seaward transport}$$

Friedrichs and Aubrey (1994) comment on the residual net transport suggesting that the direction of residual transport is, in strongly convergent channels, dominated by gradients in the current velocity alone. Areas of higher velocity, it was argued, will disperse sediment away from the area and areas of low velocity will represent a sink for sediment dispersing from areas of higher velocity. This idea produces the following criterion:

$\frac{\partial u}{\partial x} < 0 \Rightarrow$ landward transport

$\frac{\partial u}{\partial x} > 0 \Rightarrow$ seaward transport

Note that there are similarities between this criterion and the Abbott criterion relating to salinity effects if the density and saline gradients are set to zero.

In the literature there are more complex analytical formulae which consider the net residual transport resulting from fluvial flow, gravitational circulation and tidal asymmetry. However, the characterisation of estuaries as analytical or semi-analytical models is still in its infancy and the models thus far developed are not tried and tested in the wider scientific community or even, commonly, against observations. The assumptions made in the development of these models are often opaque to the reader but are very important to the validity of the resulting analytical model. It is unwise therefore to use any of these more complex analytical models in a predictive sense without further validation or alternatively a thorough background in the numerical analysis and physical understanding needed to understand and critique these models.

The complex processes of residual transport and tidal mixing are the focus of interest the investigation has essentially moved from being geomorphological to one of expert modelling and data analysis.

Conclusion

From this assessment of the analytical approach to residual sediment transport:

- It is considered that the Abbott criterion for the direction of net residual transport in estuaries with no density gradient should not be used for strongly convergent estuaries such as those commonly experienced in the UK, and instead the simpler Friedrichs and Aubrey criterion regarding the gradient of current velocity should be used.
- In estuaries where the density gradient is significant Abbott's density gradient criterion can be used.
- It is suggested that other analytical formulations for residual sediment flux are used for predictive assessment are based on experience of hydrodynamic and sediment processes, and data where possible. Also it is recommended that the relative importance of the various contributions of the respective terms in the momentum and transport equations is assessed.

Application to suspended sediment concentration profiles - theory

For estuary studies it is often the case that two main approaches to characterising suspended sediment transport are useful. The first is an initial assessment based on limited data where a "back of the envelope" type approach is required. The second is a more comprehensive study based on numerical modelling and observations to validate the numerical model. The more limited desk study approach is considered here. When using this type of approach the objectives are often to either derive the net transport through a cross-section or location or by assuming that the estuary is in balance to deduce something about the nature of the sediment transport in the estuary system. In both cases the use of sediment transport formulae is required.

One example of an approach which uses the concept of a balance between erosion and deposition is given by Soulsby (2004). Traditionally mud transport is described for modelling purposes by only four parameters: the threshold shear-stresses for erosion and deposition and the erosion-rate constant for the mud bed, and the settling velocity of the suspended mud just above the bed. Soulsby examined how these four parameters interact with the flow-generated bed shear-stresses in the simplest possible representation of a tidal estuary to determine the concentrations of mud in suspension and the masses of mud eroded and deposited per unit area.

It was assumed that the estuary was of uniform depth in both horizontal directions, and that the mud properties were horizontally uniform. The tidal depth variation is ignored, and the flow was represented by a repeating rectilinear tidal velocity such that the bed shear-stress varies sinusoidally as $\tau(t)=\hat{\tau} \,.\, \sin(\omega t)$. The settling velocity of the suspended mud was treated as constant, and the concentration profile was schematised as a linear variation from bed to surface $C(z)=C_b.(1-\alpha.z/h)$, giving a bottom concentration of $\overline{C}=\beta C_b$, where $\beta=1-\alpha/2$.

Soulsby used the idea of equilibrium to balance the pattern of erosion and deposition through the tide to derive algebraic expressions for the bottom concentration at slack water, and the maximum and minimum bottom concentrations in the tidal cycle. The maximum concentration occurs at the end of the erosion phase, and is given by:

$$C_{max} = C_o \exp(A) \qquad (28)$$

The minimum concentration occurs at the end of the deposition phase, and is given by:

$$C_{min} = C_o \exp(-A) \qquad (29)$$

Where:

$$C_o = \frac{C_s}{\sinh(A)}\left\{\left[1-\left(\frac{\tau_e}{\hat{\tau}}\right)^2\right]^{1/2} - \left(\frac{\tau_e}{\hat{\tau}}\right)\left(\frac{\pi}{2}-\phi_E\right)\right\} \qquad (30)$$

Where: $A=\dfrac{w_s\tau_d}{2\sigma\beta H\hat{\tau}}$, $C_s=\dfrac{m_e\hat{\tau}}{\sigma\beta H}$ and $\phi_E=\sin^{-1}\left(\dfrac{\tau_e}{\hat{\tau}}\right)$

The bottom concentrations C_o, C_{max}, C_{min} can be converted to depth-averaged concentrations by multiplying them by the factor β.

The parameters used in Equations 28 to 30 are listed below:

σ	Tidal radian frequency (s^{-1})
$\hat{\tau}$	Amplitude of tidal bed shear-stress ($N.m^{-2}$)
τ_e	Threshold shear-stress for erosion ($N.m^{-2}$)
τ_d	Threshold shear-stress for deposition ($N.m^{-2}$)
m_e	Mud erosion-rate constant ($kg.N^{-1}.\ s^{-1}$)
w_s	Settling velocity of flocs ($m.s^{-1}$)
β	Ratio of depth-averaged concentration to bottom concentration
C_o	Bottom concentration at slack water ($kg.m^{-3}$)
C_{max}	Maximum bottom concentration through tidal cycle ($kg.m^{-3}$)
C_{min}	Minimum bottom concentration through tidal cycle ($kg.m^{-3}$)
M_E	Mass of mud eroded per half-cycle ($kg.m^{-2}$)
M_D	Mass of mud deposited per half-cycle ($kg.m^{-2}$)

Soulsby was careful to list a number of limitations on this formulation, including:

- The flow and bed properties are assumed to be horizontally uniform, and the advection of suspended sediment is neglected, which, as noted above, is an important and often dominant effect in many estuaries.
- The concentration profile is assumed to be linear, and to change at all levels instantaneously when the bottom concentration changes.

General comments regarding sediment parameters

Although sediment parameters for non-cohesive sediments can be fairly reliably estimated, the corresponding parameters for cohesive sediment, particularly settling velocity (w_s), the erosion threshold (τ_e) and the erosion rate constant (m_e) are notoriously difficult to measure and are very site specific. The use of these parameters without proper calibration against detailed data can result in very significant error and therefore should be accompanied by an appropriate sensitivity analysis.

For non-cohesive particles (usually of diameter greater than 60 microns) the settling velocity is principally a function of the particle size. Many adequate equations to describe this relationship are described in Soulsby (1997). Cohesive sediment particles (usually of diameter less than 60 microns) in estuaries tend to aggregate together to form larger "flocs" with higher settling velocities. For cohesive sediment settling velocity is a parameter dependent on the extent of flocculation, which in turn is dependent on both the suspended sediment concentration, which affects the frequency with which sediment particles collide, and turbulence, which affects both the frequency and strength of collision, and thus can enhance flocculation (low turbulence) or reduce flocculation (high turbulence).

The erosion threshold and the rate of erosion are fairly well described for non-cohesive (sandy) sediment and formulae for the threshold of movement and the resulting transport can be derived from many well known manuals or text books (e.g. Soulsby, 1997). However, the erosion threshold and the rate of erosion for cohesive sediments is dependent on the biology and on the physico-chemical properties of the sediment in question which may vary spatially and temporally throughout an estuary or even throughout a particular mud bed. Thus any attempt to estimate these parameters carries with it a significant amount of uncertainty.

Conclusions

Based on this review of analytical approaches to estuarine sediment transport the following conclusions have been drawn:

- The equations provide good indications of the key features of the sediment transport profile. Once validated with data they might be applicable for quantitative assessments.
- For studies where a "ball-park" indication of the sediment concentration is required, and or a reasonable idea of how changes to the estuary might affect concentrations along the estuary, the Soulsby formulation can be applied. Cohesive sediment parameters, particularly settling velocity (w_s), the erosion threshold (τ_e) and the erosion rate constant (m_e) are difficult to measure and are very site specific.
- Calibrations and sensitivity tests should be carried out where possible.

Best practice in the use of analytical methods

1D estuary flow equations

The most straightforward and well-described analytical formulations for 1D flow equations are those proposed by Friedrichs and Aubrey (1994). These equations are based on a thorough evaluation of the first and second order terms and allow for the solution of cross-section averaged velocity for different estuary geometry. However, these equations are for tidally dominated estuaries and do not apply to estuaries dominated by fluvial flow.

The first order continuity and momentum equations have been presented earlier. The equations outline the hydrodynamic changes that will occur in response to changes in the estuary bathymetry and tidal range. In practice all estuaries diverge from their ideal counterparts and these analytical equations are best employed in a qualitative sense to explain how changes to a particular estuary geometry would contribute changes in tidal flows.

Residual current profiles and saline intrusion

The processes leading to residual current profiles and saline intrusion are complex and may not be clearly defined in any particular estuary. Additionally, the formulations involve "average" estuarine values for depth and current speed which, since these parameters can vary considerably along an estuary, will cause a significant source of error depending on the amount of data available and the method of calculation. Detailed assessment of residual current profiles and saline intrusion may be best undertaken as a numerical modelling study using a bottom-up process model and detailed observational data for validation. However, notwithstanding these comments it is possible to gain some qualitative insights into changes in mixing and saline intrusion by using the following relationships: for a well-mixed estuary with homogeneous lateral depth variation (Prandle, 1985);

$$u_s \sim gS_x \frac{H^2}{k\hat{u}}$$

For a well-mixed or partially mixed estuary with significant heterogeneous lateral depth variation (Fischer *et al.*, 1979):

$$u_s \sim gS_x \frac{H^2}{k\hat{u}} \left(\frac{H^2 / E_z}{W^2 / E_v} \right)$$

For a partially mixed with homogeneous lateral depth variation or stratified estuary (Prandle, 1985):

$$u_s = \frac{0.018\bar{u}}{1-d}$$ and

$$L_i \sim \frac{H^2}{f\hat{u}u_0} \frac{\Delta\rho}{\rho}$$

where: u_s is the "average" residual near bed current speed along the length of the saline intrusion due to gravitational circulation;

S_x s the density gradient ($\rho^{-1}\partial\rho/\partial x$);

H	is the "average" water depth along the length of the saline intrusion;
L_i	is the length of saline intrusion;
f & k	are coefficients with values in the region of 0.0025;
$\hat{u}$	is the "average" peak current speed along the length of the saline intrusion;
u_0	is the velocity resulting from the river discharge divided by depth (Q_f / H);
ρ & $\Delta\rho$	are the density of sea water and the difference in density between the freshwater river flow and the seawater of the marine boundary;
d (=D/H)	is the thickness of the saline lower layer as a proportion of the water depth.

Thus the following general rules can be generated:

- Residual near bed velocity caused by gravitational circulation increases with depth but reduces with width;
- Longitudinal salinity gradient but reduces with current speed;
- Length of salinity intrusion increases with depth and the salinity difference between the fresh and salt water but reduces with tidal current speed and the current speed induced by the river discharge, and friction.

Net residual sediment transport

For well-mixed strongly convergent estuaries the direction of residual transport can be is as follows (Friedrichs and Aubrey, 1994):

$$\frac{\partial u}{\partial x} < 0 \Rightarrow \text{landward transport}$$

$$\frac{\partial u}{\partial x} > 0 \Rightarrow \text{seaward transport}$$

For partially mixed estuaries the direction of residual transport is as follows (Abbott, 1960a, b):

$$\frac{1}{2} h\left(-\frac{\partial \rho}{\partial x}\right) > \left(\rho \frac{dS}{dx} + \frac{U_0}{g}\frac{\partial U_0}{\partial x}\right) \Rightarrow \text{landward transport}$$

$$\frac{1}{2} h\left(-\frac{\partial \rho}{\partial x}\right) < \left(\rho \frac{dS}{dx} + \frac{U_0}{g}\frac{\partial U_0}{\partial x}\right) \Rightarrow \text{seaward transport}$$

As stated above it is not suggested that any analytical formulations for residual sediment flux are used for predictive assessment without considerable experience of hydrodynamic and sediment processes and the relative importance of the various contributions of the respective terms in the momentum and transport equations.

The dedicated reader is invited to compare and contrast the findings of a number of relevant papers including those by Prandle referenced in this report and Schuttelaars *et al*. (2002) and Scully and Friedrichs (2002).

Suspended sediment concentration profiles

Suspended sediment transport is a complex field of study and for accurate quantitative assessment requires the application of new modelling and a considerable amount of observational data to verify the model results. However, for studies where a rough idea of the sediment concentration is required, and or a reasonable idea of how changes to the current estuary might affect concentrations along the estuary, the most straightforward approach is to balance the erosion occurring through a tide against the deposition occurring through a tide (which generally occurs at slack water). This approach is encapsulated neatly by the formulation of Soulsby (2004) outlined above.

Soulsby was careful to list a number of limitations on this formulation, which should always be considered in the context of sediment transport:

- The flow and bed properties are assumed to be horizontally uniform, and the advection of suspended sediment is neglected, which, as noted above, is an important and often dominant effect in many estuaries;
- The concentration profile is assumed to be linear, and to change at all levels instantaneously when the bottom concentration changes.

It is important to note that cohesive sediment parameters, particularly settling velocity (w_s), the erosion threshold (τ_e) and the erosion rate constant (m_e) which appear in Soulsby's formulation (and indeed any formulation of cohesive suspended sediment transport) are notoriously difficult to measure and are very site specific. The use of these parameters without proper calibration against detailed data can result in very significant error and therefore should be accompanied by an appropriate sensitivity analysis.

Conclusions

Analytical solutions are a group of mathematical expressions, often derived from basic physical principles but usually resulting from a simplification of estuary systems, that can be utilised to gain insight into the functioning and potential changes within an estuary system. Analytical solutions exist for a diverse range of physical processes and mechanisms encountered within estuarine environments, a sub-set of which have been discussed above. The role of analytical solutions is to simplify the estuary system or the physical process in question into something more tangible and useable. They are particularly useful as a conceptual aid to understanding how the system will evolve following some change or as a back-of-the envelope quantitative assessment.

However, in order to derive these analytical solutions, it is normally the case that a number of simplifying assumptions have to be made regarding the estuary system or physical process. The application of these relationships requires judgement and experience to assess whether the analytical solution is justified for the given situation.

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